SuperAlgebras

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Def A super vector space V is a vector space w/ a $\mathbb{Z}_2$-grading,

$$\overset{\text{even}}{V = V_0} \oplus \underset{}{V_1} \leftarrow \text{odd}$$

- If $\dim V_0 = m$, $\dim V_1 = n$, then up to iso, there is only one super vector space

$$\mathbb{C}^{m|n} = \overset{\text{even}}{\mathbb{C}^m} \oplus \mathbb{C}^{n} \leftarrow \text{odd} \quad \text{of } \dim(m|n)$$

- Given $a \in V_i$, $|a| = i \in \mathbb{Z}_2$

Def A superalgebra $A = \mathbb{Z}_2$-graded algebra

Concretely this means

$$A = A_0 \oplus A_1, \quad A_i A_j \subseteq A_{i+j} \quad \text{for } i,j \in \mathbb{Z}_2$$

Def A module over a superalgebra $= \mathbb{Z}_2$-graded module

Concretely this means

$$M = M_0 \oplus M_1, \quad A_i M_j \subseteq M_{i+j} \quad \text{for } i,j \in \mathbb{Z}_2$$

Ex1: Let

$$M(m|n) = \begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left(\begin{array}{cc} A & B \\ C & D \end{array}\right) \end{array} \begin{array}{l} \}m \\ \}n \end{array} = \overset{\text{even}}{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}} \oplus \overset{\text{odd}}{\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}}$$

Can check that $M(m|n)_i M(m|n)_j \subseteq M(m|n)_{i+j}$ under matrix multiplication so $M(m|n)$ is a superalg

Ex 2: Let

$$Q(n) = \begin{array}{c} \begin{array}{cc} n & n \end{array} \\ \left(\begin{array}{cc} A & B \\ B & A \end{array}\right) \end{array} \begin{array}{l} \}n \\ \}n \end{array} = \overset{\text{even}}{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}} \oplus \overset{\text{odd}}{\begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}}$$

Also a superalg under matrix multiplication

Def An ideal $I \subseteq A$ is a graded ideal if

$$I = (I \cap A_0) \oplus (I \cap A_1) \qquad (*)$$

Non-Ex: Consider two-sided ideal in $Q(n)$

$$I = Q(n) \begin{pmatrix} I & I \\ I & I \end{pmatrix} Q(n) = \left\{ \begin{pmatrix} D & D \\ D & D \end{pmatrix} \,\middle|\, D \in M_{n \times n}(\mathbb{C}) \right\}$$

Notice $I \cap Q(n)_0 = \{0\}$, $I \cap Q(n)_1 = \{0\}$, so I not graded

Def: A superalgebra A is simple if it has no nontrivial two-sided ideals.

Analogues

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Lem: $M(m|n)$ and $Q(n)$ are simple superalgs.

Pf: $M(m|n) = M_{(m+n)\times(m+n)}(\mathbb{C})$ as algs, and matrix algs are simple. For $Q(n)$, suppose $I \subseteq Q(n)$ is graded. WTS $Q(n)\, i\, Q(n) = Q(n)$ for $i \in I$. By $(*)$ can assume $i \in \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ or $i \in \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \rightsquigarrow$ reduces to $M_{n\times n}(\mathbb{C})$ $\square$

Lem: If A is a f.d. simple assoc. superalgebra

$$A \cong M(m|n) \text{ or } Q(n)$$

Thm (Super Wedderburn): A superalgebra A is semisimple (all modules are projective) $\iff$

$$A \cong \bigoplus_{i=1}^{m} M(r_i|s_i) \oplus \bigoplus_{j=1}^{q} Q(n_j)$$

aka is a $\oplus$ of simple superalgebras

- $M(m|n) \circlearrowright \mathbb{C}^{m|n}$ by matrix mult
- $Q(n) \circlearrowright \mathbb{C}^{n|n}$ ________

Lem: $\mathbb{C}^{m|n}, \mathbb{C}^{n|n}$ is a simple $M(m|n), Q(n)$ mod.

Pf: Follows from action being transitive

$$\begin{pmatrix} \boxed{A_1} & \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} A_1 \end{pmatrix}$$

complete to a matrix in $M(m|n)$ or $Q(n)$

Prop: $\mathbb{C}^{m|n}, \mathbb{C}^{n|n}$ is the unique simple $M(m|n), Q(n)$ mod

Pf: Let L be a simple $Q(n)$-mod. Simple $\Rightarrow$ $Q(n)v = L \Rightarrow$ Have SES

$$0 \to \ker \pi \to Q(n) \xrightarrow{\pi} L \to 0$$

(SW) $\Rightarrow$ seq splits so L is a summand of $Q(n)$

But $Q(n) \cong (\mathbb{C}^{n|n})^{\oplus n}$ as $Q(n)$-mod $\Rightarrow L \cong \mathbb{C}^{n|n}$

Lem (Super Schur's Lemma): If M and L are simple modules over a s.s. superalgebra A, then

$$\dim_{\mathbb{C}} \operatorname{Hom}_A(M,L) = \begin{cases} 1 & \text{if } M \cong L \text{ of type } M \\ 2 & \text{of type } Q \\ 0 & \text{if } M \not\cong L \end{cases}$$

Double Centralizer

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so wlog $K \cong L$

Pf: K, L simple $\Rightarrow 0 \neq T \in \mathrm{Hom}_A(K, L)$ must be iso

(SW) $\Rightarrow$ K and L are simples for $M(m|n)$ or $Q(n)$

(Prop) $\Rightarrow K \cong L \cong \mathbb{C}^{m|n}$ (type M) or $\mathbb{C}^{n|n}$ (type Q). Check

$$\mathrm{Hom}_{M(m|n)}(\mathbb{C}^{m|n}, \mathbb{C}^{m|n}) = \mathbb{C}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\mathrm{Hom}_{Q(n)}(\mathbb{C}^{n|n}, \mathbb{C}^{n|n}) = \mathbb{C}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Hint for ↑: When is $Q_1\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} = Q_2\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$, $Q_1, Q_2 \in Q(n)$?

- Given two superalg A, B, $A \otimes_{\mathbb{C}} B$ is a superalg

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb'$$

Lem: $Q(m) \otimes Q(n) \cong M(mn|mn)$ as superalg

Pf: dim check: $\dim Q(m) = 2m^2$, $\dim M(a|b) = (a+b)^2$

$\Rightarrow \dim Q(m) \otimes Q(n) = 2m^2 \cdot 2n^2 = 4m^2n^2$

$\dim M(mn|mn) = (2mn)^2 = 4m^2n^2$

Explicitly for $m = n = 1$, $Q(1) = \mathbb{C}e_0 \oplus \mathbb{C}e_1$, $e_1^2 = e_0$

$$\tfrac{1}{\sqrt{2}}(e_0 \otimes e_1 + e_1 \otimes e_1) \longleftrightarrow E_{12}$$

$$\tfrac{1}{\sqrt{2}}(e_1 \otimes e_0 + e_1 \otimes e_1) \longleftrightarrow E_{21}$$

$\in M(1|1)$

Cor: $\mathbb{C}^{m|m} \otimes \mathbb{C}^{n|n} \cong \mathbb{C}^{mn|mn} \oplus \mathbb{C}^{mn|mn}$ as $Q(m) \otimes Q(n)$-mod

Pf: Decompose both sides of lem

$$(\mathbb{C}^{m|m} \otimes \mathbb{C}^{n|n})^{\oplus mn} = (\mathbb{C}^{m|m})^{\oplus m} \otimes (\mathbb{C}^{n|n})^{\oplus n} = Q(m) \otimes Q(n)$$

$$\overset{\text{Lem}}{\cong} M(mn|mn) = (\mathbb{C}^{mn|mn})^{\oplus 2mn}$$

- Thus if A, B are s.s superalg, K, L simple A, B module of type Q

$\Rightarrow K \otimes L = (2^{-1} K \otimes L)^{\oplus 2}$

notation for the simple $A \otimes B$ module of type M

Thrm (Double Centralizer) Let W be a f.d. super v.s, and A a s.s subalg of $\mathrm{End}(W)$.

Let $B = \mathrm{End}_A(W)$. Then

(i) B is s.s. (ii) $\mathrm{End}_B(W) = A$

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(iii) As a $A\otimes B$-module

$$W \cong \bigoplus_i 2^{-\delta_i} V_i \otimes L_i, \quad \delta_i \in \{0,1\}$$

where $\{V_i\}$ runs over all distinct irr A-mod

$\{L_i\}$ ———————— B-mod

"Pf": (1) A s.s $\Rightarrow$ decomp of W in (iii) as A-mod

(1) + Super Schur $\Rightarrow$ B s.s. $\Rightarrow$ (ii)

Cor: $\exists$ bijection

$\{$irr rep of $A\}$ $\longleftrightarrow$ $\{$irr rep of $B\}$

2. Lie superalgebras

Def A **Lie superalgebra** is a super v/s $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ w/ bilinear map $[,] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ s.t.

(1) $[a,b] = -(-1)^{|a||b|}[b,a]$

(2) Super Jacobi identity

Rem: Given any assoc. superalg $A = A_0 \oplus A_1$ can make A into Lie superalg by setting

$$[a,b] := ab - (-1)^{|a||b|} ba$$

Ex1: (general linear Lie superalgebra)

$$\mathfrak{gl}(m|n) := (M(m|n), \quad)$$

Ex2: (queer Lie superalg)

$$\mathfrak{q}(n) := (Q(n), \quad)$$

- $\mathfrak{gl}(m|n) \curvearrowright V = \mathbb{C}^{m|n}$, then $\mathfrak{gl}(m|n) \curvearrowright V^{\otimes d}$ via

$$\Phi_d(g)(v_1\otimes\cdots\otimes v_d) = g\cdot v_1 \otimes\cdots\otimes v_d + \cdots + (-1)^{|g|(|v_1|+\cdots+|v_{d-1}|)} v_1\otimes\cdots\otimes g\cdot v_d$$

for $g \in \mathfrak{gl}(m|n)$, $v_i \in V$ homogeneous

- The symmetric group $S_d \curvearrowright V^{\otimes d}$ via

$$\Psi_d((i\ i{+}1))(v_1\otimes\cdots\otimes v_i\otimes v_{i+1}\otimes\cdots\otimes v_d) = (-1)^{|v_i||v_{i+1}|}(v_1\otimes\cdots\otimes v_{i+1}\otimes v_i\otimes\cdots\otimes v_d)$$

Schur-Sergeev

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Lem The actions of $(\mathfrak{gl}(m|n), \Phi_d)$ and (S_d, Ψ_d) on $V^{\otimes d}$ commute.

Thrm (Schur-Sergeev): (i) The images of Φ_d and Ψ_d in $\mathrm{End}_{\mathbb{C}}(V^{\otimes d})$ satisfy D.C. prop, i.e.

$$\Phi_d(U(\mathfrak{gl}(m|n))) = \mathrm{End}_{S_d}(V^{\otimes d}) \qquad (**)$$

$$\Psi_d(\mathbb{C}[S_d]) = \mathrm{End}_{U(\mathfrak{gl}(m|n))}(V^{\otimes d})$$

(ii)

$$(\mathbb{C}^{m|n})^{\otimes d} \simeq \bigoplus_{\lambda \in P_d(m|n)} L(\lambda^{\natural}) \otimes S^{\lambda}$$

Specht module

"Pf": (i) $\mathbb{C}[S_d]$ s.s $\Rightarrow \Psi_d(\mathbb{C}[S_d])$ s.s. so (i) follows if we can show $(**)$ by DC. Lem $\Rightarrow$

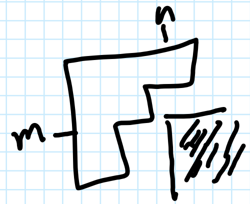
$$\Phi_d(U(\mathfrak{gl}(m|n))) \subseteq \mathrm{End}_{S_d}(V^{\otimes d})$$

$\underline{\qquad\qquad} \supseteq \underline{\qquad\qquad}$ by showing for a specific basis of "super sym poly"

(ii) Won't show, but

$P_d(m|n)$ = $(m|n)$ hook partition of d

= partition of d not containing $(m+1, n+1)$ box in Young diagram



$\square$

Rem: for $(m|0)$, recover Schur-Weyl duality

$$P_d(m|0) = \left\{ \begin{array}{l} \text{partitions } \lambda \text{ of } d \text{ s.t. } \lambda \\ \text{has at most } m \text{ parts} \end{array} \right\}$$